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Government Borrowing using Bonds with Randomly Determined Returns: Welfare Improving Randomization in the Context of Deficit Finance

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Abstract

We develop a model which formalizes a problem posed by Keynes (1940): how to optimally share the burden of deficit finance among heterogeneous agents who have differential access to inside investment opportunities. In the presence of private information, it is Pareto efficient for the government to borrow in a way that amounts to non-linear taxation, and it must treat agents with access to the best investment opportunities preferentially to keep them in the bond market. With private information about access to assets, this can often best be done by extraneously randomizing the return on the highest yielding government liabilities. The optimal government policy is shown to accord well with historical observations.

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1 Introduction

There has been increasing recent interest in trying to understand why governments finance deficits in the way they do. Our purpose in this paper is to provide a model that can simultaneously explain the following observations:[1]

(i) governments often issue an array of liabilities in different denominations and bearing different rates of return;[2]

(ii) intermediation of government (and other) liabilities is often restricted;

(iii) individuals hold diversified portfolios consisting of large denomination government bonds and divisible inside liabilities;

(iv) governments with large deficits often extraneously randomize returns on some bonds, and simultaneously issue other bonds with certain returns;

(v) some groups of agents face liquidity constraints while others do not.

In addition, our analysis bears on a problem posed in detail by Keynes (1940): How might the burden of the debt (in this case caused by World War II) be shared optimally among heterogeneous agents, some of whom have access to investments that are not available to all? We show that under plausible conditions, the answer to Keynes' question is a government borrowing scheme with all of the characteristics described above.

Interest in these issues has, of course, not been confined to the present paper. Bryant and Wallace (1984), for instance, show how a government needing to finance a deficit can improve welfare by issuing large denomination indivisible bonds and imposing legal restrictions on intermediation. This policy permits the government to employ non-linear taxation and has the implication that at an optimum all agents perceive themselves to be liquidity

[1]There is also a growing literature which employs considerations of time consistency to explain various aspects of government borrowing [e.g., Lucas and Stokey (1983), Persson, Persson, and Svensson (1987), and Chari and Kehoe (1990)]. However, we confine attention to explaining the following observations, which we believe can best be understood in contexts where government commitment is possible.

[2]Specifically, some liabilities are issued in large denominations and are indivisible.

constrained.[3] Villamil (1988) extends this analysis by incorporating heterogeneity and private information, and shows that a government behaving optimally will issue as many liabilities as there are agent types, with each liability bearing a different rate of return. Moreover, in her model only a subset of the agents face liquidity constraints.[4] However, neither model predicts the presence of non-trivially diversified individual portfolios [cf., Hoover (1988) or Villamil (1991) for criticisms of the models on this dimension] or the existence of bonds bearing extraneously randomized returns. Further, neither model permits inside investments that compete with government bonds.

In the sequel we present a model that explains each of observations (i)-(v), with particular attention paid to when randomized returns are desirable. Our interest in this topic stems from several sets of historical observations which we now describe. In the 18th century, England and France issued bonds where, in exchange for a capital payment, an investor received a title to a bond plus a lottery ticket for a drawing of additional bonds. The payoff from a "winning ticket" often provided an annual income greater than the total capital contributed. The capital contributed generally was not small, sometimes exceeding average per capita income [see Weir and Velde (1989)]. Thus, we observe large denomination bonds for which the return is explicitly extraneously randomized. At this same time England and France also borrowed heavily through the use of tontines, where a group of subscribers purchased bonds with fixed payments divided among "survivors." With a large group of subscribers, the government's payments displayed little randomness, but for any individual the returns were random. Interestingly, these were not simple annuity schemes since subscribers could make the payment contingent

[3]The optimal policy has a "forced saving" aspect which each agent would like to undo. This is prevented by restrictions on intermediation which agents view as liquidity constraints on borrowing.

[4]Having liquidity constraints perceived by only a subset of agents is consistent with empirical evidence on credit rationing [cf., Jappelli (1990)].

on the survival of someone other than themselves.[5] The expected returns on this type of liability were generally favorable [see Weir (1989)]. Finally, during the American Revolution the Continental government attempted to borrow, in Europe, through the use of so-called lottery bonds with randomly determined returns [see Anderson (1982)]. This was viewed as a device for making American debt instruments more attractive to European investors.[6]

The use of such debt instruments has often been viewed as puzzling because the necessity of paying a risk premium to compensate (presumably) risk averse borrowers for randomized returns makes this an apparently expensive way to borrow [see the discussion in Weir (1989)]. Furthermore, in the episodes described, financial and insurance markets were at best weakly developed. Therefore, one might expect the government to be relatively better able to bear risk than individuals, so the *intentional* introduction of extrinsic uncertainty by governments merits explanation. Also, "lottery bonds" and bonds with certain returns were often used simultaneously. Finally, in some of the historical examples, bonds with randomized returns were intended to be sold to relatively wealthy investors. This contrasts with the socio-economic characteristics of participants in recent state sponsored lottery "games" [see Clotfelter and Cook (1990)]. These observations merit explanation.

We develop a stationary, two-period lived overlapping generations model

[5] Weir and Velde (1989) describe one famous mechanism, the "thirty French girls," that was very transparent. Lists of young girls from Genevan families with reputations for longevity and who had survived smallpox were compiled for use as "nominees" (recall that subscribers could make payments contingent on the survival of someone other than themselves, but payment required proof of the nominees' survival). The most common group size was thirty because administrative costs rose with the number of nominees and the marginal reduction in "portfolio risk" became small after thirty.

[6] We defer discussion of World War II finance, the problem considered by Keynes, to Section 5. Also, we note that bond returns are often randomized in practice by fixing a nominal payment in the presence of stochastic inflation rates or by simple (partial) defaults. Both of these randomization schemes can be explained by our analysis. However, our primary interest is in understanding the explicit randomization schemes described in the text, which were well understood by agents to be random and were voluntarily entered into.

in which a government with a utilitarian social welfare function must finance a fixed deficit of a given size. It does this by borrowing from two types of agents that are identical in all respects but one: different agents have different access to investment opportunities other than government bonds. This is intended to capture a situation where wealthier investors have access to investment opportunities not open to poorer investors, or where a government seeks to borrow both at home and abroad, and foreign investors have investment opportunities not open to domestic investors. We assume that agent type and outside investment activity are private information. If agents did not have differential access to outside investment opportunities the government would raise revenue from all types equally. When the deficit is sufficiently large, doing so drives agents with the best investment opportunities (say type 1 agents) out of the bond market. This, in turn, requires all revenue to be raised from type 2 agents, which a utilitarian government regards as undesirable. Thus the government raises as much revenue as it can from type 1 agents without driving them out of the bond market, and this requires that type 1 agents be treated preferentially. However, when type is private information, preferential treatment of type 1 agents creates an adverse selection problem, and optimal government policy must address the problem of keeping type 1 agents in the bond market in the face of this problem.

Under conditions we describe, the government treats type 1 agents preferentially by designing an asset for them with a randomized return. In contrast, the asset designed for type 2 agents has a lower expected (but certain) return. Despite the fact that both agent types have identical preferences, endowments, and equal access to the government's assets, the access of type 1 agents to an outside alternative allows them to partially insure against the bad state of nature associated with the randomized return. Type 2 agents, having no access to the outside alternative, prefer the certain return. The solution to this problem also involves the use of a kind of price discrimination described by Bryant and Wallace (1984) or Villamil (1988), where (as in

Villamil) government liabilities are issued in minimum denominations, intermediation is prohibited, and there are as many types of government bonds (bearing different returns) as there are agent types. In addition, type 1 agents are treated preferentially because they have access to private investment opportunities. We show that this policy is constrained Pareto efficient if absolute risk aversion decreases at a rapid enough rate, because it is then the optimal way to keep type 1 agents in the bond market given the adverse selection problem.

The result that bonds with extraneously randomized returns are constrained Pareto efficient can also be interpreted as asserting the desirability of random taxation. Of course the potential desirability of random taxation in the presence of an adverse selection problem has been previously pointed out [for instance by Stiglitz (1982)]. In our analysis, however, where the focus is on government bond sales, the government cannot compel participation. This makes our analysis somewhat different from that in standard taxation models. Interpreted in a taxation context, our model could be regarded as one in which only market activities can be taxed, and high enough taxation drives some agents into non-market (or "underground") activities. Thus, taxation must not only raise sufficient revenue, it must be designed to prevent exit from market activities. Note that the adverse selection problem arises in this setting if and only if voluntary participation is a binding constraint.

Finally, we briefly relate our results to two other literatures. First, in our model (non-optimal) extraneous uncertainty can be introduced by market factors which allow sunspot equilibria to exist, as in Shell (1977), Azariadis (1981), or Cass and Shell (1983). However, our focus is different; we describe conditions under which the government will intentionally, and on welfare grounds, inject extraneous uncertainty into allocations. Second, in adverse selection models where agents have identical underlying preferences, it is generally not the case that randomization of allocations is desirable [e.g., see Prescott and Townsend (1984) or Arnott and Stiglitz (1988)]. In contrast, we

show that randomization is desirable in an environment where agents have identical underlying preferences and endowments, because different agent "types" have differential access to outside opportunities. We return to the relationship between our work and these literatures in the final section.

The remainder of the paper proceeds as follows. Section 2 describes the model. Section 3 considers non-stochastic planning problems from which Pareto efficient consumption allocations can be derived under three alternative sets of assumptions about the constraints faced by the planner. Section 4 establishes conditions under which randomized allocations are desirable. In both Sections 3 and 4 we describe how the government can decentralize the optimal allocations. Section 5 relates our analysis to the problem posed by Keynes (1940), while Section 6 concludes.

2 The Model

Consider a discrete time economy populated by an infinite sequence of two-period lived, overlapping generations and an infinitely lived government. Each generation is identical in size and composition, containing a continuum of agents with unit mass. Within each generation there are two types of agents, indexed by $i = 1, 2$. Let θ_i denote the fraction of type i agents in each generation, with $\theta_i > 0$ and $\theta_1 + \theta_2 = 1$. In addition, there is a single consumption good at each date. All agents have endowment w_j of the good in period $j = 1, 2$ of their life, with $w_j \geq 0$.

Agent types are differentiated by their access to a storage technology. Type 1 (and only type 1) agents have access to a constant returns to scale technology for storing the good, where one unit stored at time t returns $x \in (0, 1)$ units at time $t + 1$. Assume that each agent can store only his or her own good, that agent type is private information (ex-ante), and that the activity of storing the good (or the quantity stored) is unobservable.

All agents have identical preferences, representable by the additively sep-

arable utility function $u(c_1^i)+v(c_2^i)$, where $c_j^i \in I\!R_+$ denotes the consumption of a type i agent at age j. Assume that u and v are strictly increasing, strictly concave, and thrice continuously differentiable, and define $R(c) \equiv -\frac{v''(c)}{v'(c)}$ to be the coefficient of absolute risk aversion. Finally, let the government have an exogenously given real per capita expenditure level of $g > 0$ each period. Assume that agents derive no utility from this expenditure.[7]

The assumption that type 1 agents can store the good while type 2 agents cannot is meant to capture the problem facing a government which has a deficit to finance, and must borrow from a set of heterogeneous agents with differential access to alternative investment opportunities. The access of some agents to relatively high return investments limits the ability of the government to extract resources from them. In the model, the ability of type 1 agents to store the good gives them access to an asset not available to type 2 agents, which is the simplest form this problem can take. It proxies for several different scenarios. For example, one might imagine that wealthier agents have access to investment opportunities not available to poorer agents.[8] Alternatively, it could represent the situation of a domestic government which seeks to borrow from foreign investors, who have investment options (bearing the gross rate of return x) not available to domestic investors.[9] Finally, we note that our model has an interpretation as an economy in which direct taxation is employed, but only "market activities" can be taxed. In this case the differential access to the storage technology proxies for different "non-market" opportunities.

For future reference, it will be useful to have a notation for the savings

[7] Or, government expenditure could affect utility in an additively separable way.

[8] This is the problem considered by Keynes (1940), which is discussed further in Section 5. Formally, we could let type i agents have an age j endowment of w_j^i, with $w_1^1 > w_1^2$, which resembles the model of Villamil (1988). However, this formulation complicates the analysis without adding additional substantive issues. Therefore, we do not pursue it here.

[9] This would represent the problem facing the American government during the Revolution when it contemplated selling lottery bonds in Europe.

behavior of an agent who pays a lump-sum tax of τ_j at age j, and faces a certain gross rate of return on savings of r. Such an agent chooses a savings level, q, to maximize $u(w_1 - \tau_1 - q) + v(w_2 - \tau_2 + rq)$ subject to non-negativity constraints. The solution to this problem is given by the savings function $q \equiv f(w_1 - \tau_1, w_2 - \tau_2, r)$. Under our assumptions, and assuming interiority, $f_1 > 0 > f_2$. Also, we assume that

$$w_1 > f(w_1, w_2, x) > 0. \tag{a.1}$$

Finally, we define the indirect utility function V in the standard way:

$$V(w_1 - \tau_1, w_2 - \tau_2, r) \equiv u(w_1 - \tau_1 - f(\cdot)) + v(w_2 - \tau_2 + rf(\cdot)).$$

3 Non-random Pareto Efficient Allocations

This section describes the allocations which solve a utilitarian social welfare maximization problem under three alternative sets of assumptions about constraints faced by a social planner. It then considers how these allocations can be decentralized by a government which sells bonds competitively, but can impose legal restrictions on bond trades. Under each set of assumptions about the constraints, we comment on which of observation (i) through (v) are captured by the analysis. Finally, in this section attention is restricted to non-random consumption allocations.

3.1 Full Information

As a benchmark, we begin by considering the problem of a social planner under full information; that is, we assume that the planner knows each agent's type, and can observe and (if desired) prohibit storage of the good. The planner's objective is to find a stationary allocation that maximizes an equally weighted sum of the agents' utilities subject to a resource feasibility constraint. Let k denote the amount of storage by a type 1 agent. Then the full information Pareto problem can be written as follows:

Problem 3.1. *For $i = 1, 2$, choose values c_1^i, c_2^i and k to maximize:*

$$\sum_{i=1}^{2} \theta_i [u(c_1^i) + v(c_2^i)]$$

subject to:

$$\sum_{i=1}^{2} \theta_i (c_1^i + c_2^i) + \theta_1 k \leq w_1 + w_2 - g + \theta_1 x k. \tag{1}$$

At an interior optimum, the solution to this problem sets

$$u'(c_1^i) = v'(c_2^i), \tag{2}$$

for $i = 1, 2$, $c_j^1 = c_j^2$, for $j = 1, 2$, and $k = 0$. Notice that, from equations (1) and (2), $c_1^i = w_1 - f(w_1, w_2 - g, 1)$, and $c_2^i = w_2 - g + f(w_1, w_2 - g, 1)$. Thus the utility of agents under this allocation is given by $V(w_1, w_2 - g, 1)$.

Remark. The allocation given by the solution to Problem 3.1 is identical to that obtained by Bryant and Wallace (1984), and can be decentralized as they describe. In particular, the government can prohibit goods storage, sell bonds with a minimum real value of F and a rate of return r, and prohibit agents from "sharing" (or intermediating) bonds. If F and r are chosen to satisfy $F = f(w_1, w_2 - g, 1)$ and $r = \frac{F-g}{F}$, then each agent will voluntarily purchase bonds with a minimum real value of F [when $V(w_1, w_2 - g, 1) \geq V(w_1, w_2, 0)$]. This policy permits the government to raise enough revenue to cover its expenditure.

This arrangement has the feature that the government sells indivisible, large denomination bonds. In addition, each agent saves more than he or she would prefer at the going rate of return in a market without restrictions on intermediation. Specifically, all individuals would like to borrow against the future income from their investments (to consume more now), but are precluded from doing so by legal restrictions. Thus, contrary to observation (v), *all* agents perceive themselves as liquidity constrained. In other words, this

model [which is simply Bryant and Wallace (1984)] captures the observation that governments sell large denomination bonds and restrict intermediation (i.e., observation (i) and (ii)), but it captures none of the other observations listed in the Introduction.

3.2 Voluntary Participation

We now assume that the planner is subject to a voluntary participation constraint, or in other words, that the planner cannot prevent type 1 agents from autarchically storing the good or type 2 agents from consuming their endowments. This represents the situation of a government that must finance a deficit g by selling bonds, where the government is unable to compel bond purchases. Alternatively, we may view this as the situation of a government that cannot tax activities in an "underground" economy. We continue to assume that the government observes agents' types directly.

The planner now solves the problem

Problem 3.2. *For $i = 1, 2$, choose c_1^i, c_2^i, and k to maximize:*

$$\sum_{i=1}^{2} \theta_i [u(c_1^i) + v(c_2^i)]$$

subject to: (1) and

$$u(c_1^1) + v(c_2^1) \geq V(w_1, w_2, x); \tag{3}$$

$$u(c_1^2) + v(c_2^2) \geq u(w_1) + v(w_2). \tag{4}$$

There are three possibilities regarding the solution to Problem 3.2.

Case 1: $V(w_1, w_2 - g, 1) \geq V(w_1, w_2, x)$. In this case constraints (3) and (4) do not bind. This occurs, obviously, if g is sufficiently small, in which case the solution to Problem 3.2 is the same as the solution to Problem 3.1.

Case 2: $V(w_1, w_2 - g, 1) < u(w_1) + v(w_2)$. In this case the constraint set is empty. We henceforth abstract from this possibility, which occurs if g is too large.

Case 3: $V(w_1, w_2, x) > V(w_1, w_2 - g, 1) \geq u(w_1) + v(w_2)$. In this case constraint (3) binds. This is the situation of interest to us and we therefore focus exclusively on it. The solution satisfies constraints (1) and (3) as equalities, (2), and $c_j^1 > c_j^2$, for $j = 1, 2$. In addition, $k = 0$. Thus, due to the government's inability to compel agents to purchase its bonds, type 1 agents must be given incentives not to withdraw from the bond market. Consequently, type 1 agents receive better terms than type 2 agents. Notice, however, that since (2) holds, no inefficiencies result.[10]

Remark. The allocation given by the solution to Problem 3.2 can be decentralized by the following government policy. Bonds are sold to type i agents with a minimum real value of F^i and a corresponding gross rate of return r^i. Then $F^i = w_1 - c_1^i$ and $r^i = \frac{c_2^i - w_2}{F^i}$ hold. Type 2 agents are prohibited from buying type 1 bonds, and intermediation is prohibited ex cathedra. Arguments following those of Bryant and Wallace (1984) establish that type i agents voluntarily purchase F^i units of bonds of type i. It is easy to verify that this permits the government to raise revenues equal to its expenditures.

This arrangement has all of the features of the problem in Section 3.1. Relative to the former arrangement, however, it indicates that the government now issues many types of bonds bearing alternative rates of return. However, there is still no diversification of individuals' portfolios.

3.3 Voluntary Participation and Private Information

We next consider the problem of a planner who wishes to choose non-stochastic Pareto efficient consumption allocations but cannot compel market participation, and in addition, cannot directly observe the type of any agent. Thus, the planner is subject to incentive compatibility constraints, as well as the other constraints specified previously.

The planner now solves the problem

[10] This corresponds to first degree price discrimination (or lump sum taxation).

Problem 3.3. *For $i = 1, 2$, choose c_1^i, c_2^i and k to maximize*

$$\sum_{i=1}^{2} \theta_i [u(c_1^i) + v(c_2^i)]$$

subject to: (1), (3), (4), *and the self-selection constraints*

$$u(c_1^1) + v(c_2^1) \geq u(c_1^2) + v(c_2^2); \tag{5}$$

$$u(c_1^2) + v(c_2^2) \geq u(c_1^1 + k) + v(c_2^1 - xk). \tag{6}$$

Equation (5) imposes that type 1 agents (weakly) prefer (c_1^1, c_2^1) to (c_1^2, c_2^2).[11] Equation (6) imposes incentive compatibility for type 2 agents, since a type 2 agent taking a type 1 allocation is not able to mimic the storage of a type 1 agent. Thus such an agent consumes $c_1^1 + k$ when young, and $c_2^1 - xk$ (i.e., c_2^1 less the proceeds of storage) when old.

The solutions to Problem 3.3 are of two general types.

Case 1: $V(w_1, w_2 - g, 1) \geq V(w_1, w_2, x)$. In this case the allocation from Problem 3.1 satisfies conditions (3) through (6), since $c_j^1 = c_j^2$, for $j = 1, 2$.

Case 2: $V(w_1, w_2, x) > V(w_1, w_2 - g, 1)$. In this case the allocation from Problem 3.2 clearly is not incentive compatible, since $c_j^1 > c_j^2$, for $j = 1, 2$, and $k = 0$. In particular, since there is no goods storage and type 1 agents are "better treated" than type 2 agents, all type 2 agents will claim to be of type 1. We now focus on this case. It is clear that if (3) holds with equality, then (5) will be satisfied. Hence (1), (3), and (6) are the binding constraints in Problem 3.3. Moreover, the constraint set will be non-empty if, for instance, $V(w_1, w_2 - \frac{g}{\theta_2}, 1) \geq u(w_1) + v(w_2)$ holds.

We now characterize the solution to Problem 3.3.

Proposition 1. *The solution to Problem* 3.3 *satisfies* (1), (3), *and* (6) *at equality. In addition, it has $u'(c_1^1) = xv'(c_2^1)$, $u'(c_1^2) = v'(c_2^2)$, and $k > 0$.*

[11] Formally, constraint (5) should be written as $u(c_1^1) + v(c_2^1) \geq V(c_1^2, c_2^2, x)$. However, since $u'(c_1^2) = v'(c_2^2) > xv'(c_2^2)$ holds (see below), $V(c_1^2, c_2^2, x) = u(c_1^2) + v(c_2^2)$.

Proof. See Appendix A.

Remark 1. The solution to Problem 3.3 has at least two interesting features. First, as Appendix A shows, goods storage occurs. This is necessary to give type 1 agents a utility level of $V(w_1, w_2, x)$ without having type 2 agents mimic their bond purchases. Second, since $u'(c_1^1) = xv'(c_2^1)$, type 1 agents are "on their savings functions" with respect to storage of the good. Both of these features reflect inefficiencies due to the necessity of treating type 1 agents preferentially in the presence of private information.

Remark 2. The solution to Problem 3.3 can be decentralized as follows. The government issues two types of bonds, and prevents agents from sharing them. Agents who buy type 1 bonds can buy only type 1 bonds, and are permitted to purchase *at most* $\bar{F}$ units (in real terms). These bonds earn the gross return r^1. Agents who purchase type 2 bonds must purchase at least F^2 units (in real terms), and these bonds earn the gross return r^2. The government chooses F^2 and r^2 to satisfy $F^2 = w_1 - c_1^2$ and $r^2 = \frac{c_2^2 - w_2}{F^2}$. The government chooses $\bar{F}$ and r^1 to satisfy

$$c_1^1 = w_1 - f(w_1 - \bar{F}, w_2 + r^1\bar{F}, x); \tag{7}$$

$$c_2^1 = w_2 + r^1\bar{F} + xf(w_1 - \bar{F}, w_2 + r^1\bar{F}, x). \tag{8}$$

Then by (7) and (8), type 1 agents are "on their savings functions." Type 2 agents optimally purchase F^2 units of type 2 bonds, and the government raises revenue with a per capita value of g.

Remark 3. This arrangement captures observations (i) and (ii). In addition, it has the feature that type 1 agents hold diversified portfolios, since they store goods and hold government bonds. Moreover, type 1 agents are "on their savings functions," and hence do not perceive themselves to be "liquidity constrained." Type 2 agents do, since $u'(c_1^2) = v'(c_2^2) > r^2v'(c_2^2)$.

Thus, this context explains why government imposed legal restrictions lead some, but not all, agents' to perceive themselves to be liquidity constrained. The only remaining observation to be confronted then, is that governments sometimes simultaneously issue bonds with randomized returns and bonds with non-randomized returns [observation (*iv*)].

4 Stochastic Pareto Efficient Allocations

We now state conditions under which extrinsic randomization can be Pareto improving. We begin by introducing some notation, and then consider a constrained social planning problem that permits extraneous uncertainty. Our objective is only to show that some extrinsic randomization is desirable, so we proceed as follows. We assume that the planner chooses, for $i = 1, 2$, deterministic values c_1^i for young consumption, and values $c_2^i(s)$ for old consumption that may depend on an extraneous state s. For simplicity we let $s \in \{1, 2\}$, and let $p \in (0, 1)$ be the exogenous probability (which is the same in all periods) that $s = 1$.[12] We assume that realizations of s are independently and identically distributed across agents, and that s is realized at the beginning of old age. This is meant to capture the features of several historical randomization devices employed in government borrowing. As we noted in the Introduction, historical evidence indicates that governments have confronted individuals with random returns on some bonds, while the government faced little or no (as here) randomness with respect to interest obligations on these bonds.

We now consider the planning problem from which constrained, Pareto efficient (possibly stochastic) consumption allocations are chosen. To simplify notation, we will sometimes write $E_s h(c_2^i(s)) \equiv ph(c_2^i(1)) + (1-p)h(c_2^i(2))$, where $h(\cdot)$ is an arbitrary function, and E denotes the expectation operator.

[12]While our description treats p as exogenous, clearly randomization can be no less desirable on welfare grounds if the government is free to choose p.

The stochastic Pareto problem can be written as follows.

Problem 4.1. *For $i = 1, 2$ and $s = 1, 2$, choose c_1^i, $c_2^i(s)$, and k to maximize*

$$\sum_{i=1}^{2} \theta_i [u(c_1^i) + E_s v(c_2^i(s))]$$

subject to:

$$\sum_{i=1}^{2} \theta_i [c_1^i + E_s c_2^i(s)] + \theta_1 k \leq w_1 + w_2 - g + \theta_1 x k; \tag{9}$$

$$u(c_1^1) + E_s v(c_2^1(s)) \geq V(w_1, w_2, x); \tag{10}$$

$$u(c_1^2) + E_s v(c_2^2(s)) \geq u(c_1^1 + k) + E_s v(c_2^1(s) - xk); \tag{11}$$

$$u(c_1^1) + E_s v(c_2^1(s)) \geq u(c_1^2) + E_s v(c_2^2(s)); \tag{12}$$

$$u(c_1^2) + E_s v(c_2^2(s)) \geq u(w_1) + v(w_2). \tag{13}$$

Equation (9) is the resource feasibility constraint, which reflects the fact that there is no aggregate randomness. Equations (11) and (12) are the self-selection constraints, and equations (10) and (13) are the voluntary participation constraints.

Clearly the solution to Problem 4.1 coincides with the solution to Problem 3.1 unless (10) is binding. When (10) binds, so does (11), as in the previous section. In this case, (12) cannot bind, and we restrict attention to the case in which (13) does not bind. Thus, for the remainder of the section, constraints (9) through (11) bind. It is easy to verify in this case that the solution to Problem 4.1 has $c_2^2(1) = c_2^2(2)$, so that only (or at most) type 1 agents face extrinsic uncertainty. This captures the observation that historically governments with large deficits have made use of lottery bonds with attractive return distributions that are sold to agents who (presumably) have reasonable alternative investment opportunities. Also, it is easy to check that

the solution to Problem 4.1 has $u'(c_1^2) = v'(c_2^2)$. Finally, arguments identical to those in Appendix A can be used to establish that $k > 0$ holds, and that

$$u'(c_1^1) = xE_s v'(c_2^1(s)). \tag{14}$$

Thus, type 1 agents are "on their savings functions," as before.

4.1 Welfare Improving Randomization

We now state a sufficient condition for $c_2^1(1) \neq c_2^1(2)$ to hold, so that type 1 agents face extraneous uncertainty.

Proposition 2. *Suppose that* $-\frac{(1-x)v''(\tilde{c}_2^1 - x\tilde{k})}{v'(\tilde{c}_2^1 - x\tilde{k}) - u'(\tilde{c}_1^1 + \tilde{k})} > -\frac{v''(\tilde{c}_2^1)}{v'(\tilde{c}_2^1)}$ *holds, where* $\tilde{c}_1^1$, $\tilde{c}_2^1$, *and* $\tilde{k}$ *are solutions to Problem* 3.3. *Then* $c_2^1(1) \neq c_2^1(2)$.

Proof. See Appendix B.

In the remainder of this section, we provide necessary and sufficient conditions for the inequality in Proposition 2 to hold. We then interpret our result. In particular, we find that when the elasticity of absolute risk aversion with respect to old age consumption is sufficiently large, extrinsic randomization is welfare improving.

The inequality in Proposition 2 can be rewritten

$$\frac{R(\tilde{c}_2^1 - x\tilde{k})(1-x)v'(\tilde{c}_2^1 - x\tilde{k})}{v'(\tilde{c}_2^1 - x\tilde{k}) - u'(\tilde{c}_1^1 + \tilde{k})} > R(\tilde{c}_2^1). \tag{15}$$

Note that $\frac{v'(\tilde{c}_2^1 - x\tilde{k})}{v'(\tilde{c}_2^1 - x\tilde{k}) - u'(\tilde{c}_1^1 + \tilde{k})} < \frac{1}{1-x}$, so a necessary condition for (15) to hold is decreasing absolute risk aversion (with respect to old age consumption). We now derive a sufficient condition for (15) to hold.

Define the function G by

$$G(\tilde{c}_1^1, \tilde{c}_2^1, k) \equiv \frac{R(\tilde{c}_2^1 - xk)(1-x)v'(\tilde{c}_2^1 - xk)}{v'(\tilde{c}_2^1 - xk) - u'(\tilde{c}_1^1 + k)} - R(\tilde{c}_2^1). \tag{16}$$

Since $(\tilde{c}_1^1, \tilde{c}_2^1)$ satisfies $u'(\tilde{c}_1^1) = xv'(\tilde{c}_2^1)$, it follows that $G(\tilde{c}_1^1, \tilde{c}_2^1, 0) = 0$. Moreover, $u(\tilde{c}_1^1) + v(\tilde{c}_2^1) = V(w_1, w_2, x)$ holds, so that $(\tilde{c}_1^1, \tilde{c}_2^1)$ is completely determined and independent of k. Thus, G is effectively a function of k alone, and if $G_3(\tilde{c}_1^1, \tilde{c}_2^1, k) > 0$ for all $k > 0$, $G(\tilde{c}_1^1, \tilde{c}_2^1, \tilde{k}) > 0$ will hold. This, of course, is exactly (15).

Straightforward differentiation of (16) establishes that $G_3 > 0$ iff

$$-\frac{xR'(\tilde{c}_2^1 - xk)}{R(\tilde{c}_2^1 - xk)} > -\{\frac{u'(\tilde{c}_1^1 + k)}{v'(\tilde{c}_2^1 - xk) - u'(\tilde{c}_1^1 + k)}\}\times$$

$$\{\frac{u''(\tilde{c}_1^1 + k)}{u'(\tilde{c}_1^1 + k)} - xR(\tilde{c}_2^1 - xk)\}. \tag{17}$$

Since $xv'(\tilde{c}_2^1 - xk) \geq u'(\tilde{c}_1^1 + k)$ for all $k \geq 0$, a sufficient condition for $G_3(\tilde{c}_1^1, \tilde{c}_2^1, k) > 0$, for all $k \geq 0$, is

$$-\{\frac{(1-x)R'(\tilde{c}_2^1 - xk)}{R(\tilde{c}_2^1 - xk)}\} \geq xR(\tilde{c}_2^1 - xk) - \frac{u''(\tilde{c}_1^1 + k)}{u'(\tilde{c}_1^1 + k)}. \tag{18}$$

An alternative statement of (18) is obtained by multiplying both sides by $(\tilde{c}_2^1 - xk)$ to get

$$-\{\frac{(1-x)R'(\tilde{c}_2^1 - xk)(\tilde{c}_2^1 - xk)}{R(\tilde{c}_2^1 - xk)}\} \geq xR(\tilde{c}_2^1 - xk)(\tilde{c}_2^1 - xk) -$$

$$\{(\frac{\tilde{c}_2^1 - xk}{\tilde{c}_1^1 + k})\frac{u''(\tilde{c}_1^1 + k)(\tilde{c}_1^1 + k)}{u'(\tilde{c}_1^1 + k)}\}. \tag{18'}$$

Equation (18′) provides the result. It asserts that $G_3(\tilde{c}_1^1, \tilde{c}_2^1, k) > 0$ for all $k \geq 0$ if the elasticity of absolute risk aversion [with respect to old age consumption, $\frac{R'(\cdot)(\cdot)}{R(\cdot)}$], is sufficiently large. Note that $R'(\cdot) < 0$ if the utility function exhibits everywhere strictly decreasing absolute risk aversion.[13]

[13]Decreasing absolute risk aversion implies that in a choice between a safe and a risky asset, the risky asset is a normal good. This is a common assumption about preferences.

4.2 A Special Case

We now consider the special case in which $u(c_1) = \phi\frac{c_1^{1-\rho}}{1-\rho}$ and $v(c_2) = \frac{c_2^{1-\rho}}{1-\rho}$, with $\phi \geq x$ and $\rho > 0$. Then $-\frac{cR'(c)}{R(c)} \equiv 1$ for all c. In addition, $u'(\tilde{c}_1^1) = xv'(\tilde{c}_2^1)$ implies that $\tilde{c}_2^1 = \tilde{c}_1^1(\frac{x}{\phi})^{1/\rho} \leq \tilde{c}_1^1$, and consequently, that $\tilde{c}_2^1 - xk \leq (\tilde{c}_1^1 + k)(\frac{x}{\phi})^{1/\rho}$ for all $k \geq 0$. In this case (18′) reduces to

$$1 - x \geq x\rho + \rho\frac{(\tilde{c}_2^1 - xk)}{(\tilde{c}_1^1 + k)} \tag{19}$$

for all $k \geq 0$, which of course holds if

$$1 - x \geq x\rho + \rho(\frac{x}{\phi})^{1/\rho}. \tag{20}$$

Thus $G_3(\tilde{c}_1^1, \tilde{c}_2^1, k) > 0$ for all $k \geq 0$ holds if ρ is sufficiently small. This, in turn, implies that the inequality in Proposition 2 holds, and that randomization is desirable on welfare grounds.

4.3 Decentralizing the Optimal Stochastic Allocation

We must slightly augment our notation from Section 2 in order to describe how to decentralize the optimal stochastic allocation. Consider the savings problem of a young agent who faces a random lump-sum tax of $\tau_2(s)$ when old, $s = 1, 2$, where the probability that $s = 1$ is p, and who faces a deterministic gross rate of return r. This agent's problem is to choose a savings level q to maximize $u(w_1 - \tau_1 - q) + pv(w_2 - \tau_2(1) + rq) + (1-p)v(w_2 - \tau_2(2) + rq)$. The solution to the problem is a savings function $q \equiv \tilde{f}(w_1 - \tau_1, w_2 - \tau_2(1), w_2 - \tau_2(2), r; p)$.

The optimal random consumption allocation can be supported by the following policy. The government sells two types of bonds, and imposes restrictions which prohibit agents from sharing bonds. The bonds sold to type 2 agents are sold in a minimum denomination of F and bear a deterministic return r. The government chooses F and r to satisfy $c_1^2 = w_1 - F$ and

$c_2^2 = w_2 + rF$. The bonds sold to type 1 agents are sold only in the indivisible amount $\hat{F}$, and bear a gross return $\hat{r}(1)$ with probability p, and $\hat{r}(2)$ with probability $1-p$. The government chooses $\hat{F}$, $\hat{r}(1)$, and $\hat{r}(2)$ to satisfy

$$c_1^1 = w_1 - \tilde{f}[w_1 - \hat{F}, w_2 + \hat{r}(1)\hat{F}, w_2 + \hat{r}(2)\hat{F}, x; p]; \tag{21}$$

$$c_2^1(1) = w_2 + \hat{r}(1)\hat{F} + x\tilde{f}(\cdot); \tag{22}$$

$$c_2^1(2) = w_2 + \hat{r}(2)\hat{F} + x\tilde{f}(\cdot). \tag{23}$$

This construction works since, by (14), type 1 agents are "on their savings functions."

This decentralization scheme yields all of the observations listed in the Introduction. Bonds are sold in many indivisible denominations with alternative rates of return, and intermediation of bonds is restricted. There is individual portfolio diversification by type 1 agents, who store the good and hold government bonds. Moreover, these agents do not perceive liquidity constraints, while type 2 agents do. Finally, the government simultaneously issues bonds with randomized returns and bonds with certain returns.

5 Some Notes on a Problem of Keynes

As mentioned at the outset, the problem we address bears a strong resemblance to that posed by Keynes (1940). Writing at the beginning of World War II, Keynes argued that it was not desirable (and possibly not feasible) to finance large wartime government expenditures without running a deficit. He also argued that simple monetization of the deficit (without legal restrictions) would result in an inflation tax which would be inefficient. Moreover, he concluded that the inflation tax could largely be evaded by the "entrepreneur class" (p. 29), which had access to real assets. Hence, the burden of the tax would fall entirely on the poor (whose access to assets was limited). The problem, then, was how to finance the deficit in a way that would be

least burdensome to the poor, given that there was a limit to the amount of resources that could be raised from the wealthy.

The solution Keynes proposed was a "forced savings program," in which agents were obligated to buy bonds. Interestingly, he also proposed a stochastic element to repayment, which was that repayment of the debt would begin in the first postwar cyclical downturn.[14] Thus, the timing of repayment was clearly random from the point of view of bondholders. Sargent (1987) describes the analysis of Bryant and Wallace (1984) as a solution to Keynes' problem. However, they consider an economy with no heterogeneity, so they show only that a forced savings program can reduce the inefficiency of an inflation tax.[15] However, they do not address the problem of the distribution of the burden associated with deficit finance [i.e., the classic vertical equity or "ability to pay" problem in public finance]. This was Keynes' primary interest, and in fact he explicitly argued that, in the absence of heterogeneity, the problem would be trivial, and could be solved by rationing (p. 52).

We explicitly confront Keynes' problem, in that our model has two "classes," one of whom has access to real investments not available to the other. We then are able to show that the solution to the problem is to institute a forced savings program, where the quantity of forced savings differs for different groups. Further, we show that randomization of returns is a desirable aspect of such a scheme under plausible conditions.

6 Conclusions

We have described an environment in which a government must finance a fixed deficit of a given size. When some agents have access to investment

[14]Obviously, Keynes intended that this would be an expansionary policy measure.

[15]Specifically, the government reduces the inefficiency of the tax by issuing bonds with a higher yield (than currency) and by imposing a set of legal restrictions that cause "forced savings" and thereby expand the tax base.

opportunities other than government bonds, government borrowing is constrained by the desirability of keeping these agents in the bond market. However, treating some agents preferentially creates an adverse selection problem. The optimal solution to these two problems involves price discrimination by the government, and may involve the simultaneous use of bonds with random and non-random returns. Interestingly, agents with the best outside investment opportunities purchase bonds with random returns, and extraneous randomization of bond returns is observed only when the government's revenue needs are sufficiently large. These two features accord well with the historical observations cited in the Introduction. This borrowing mechanism can also be interpreted as one in which inflation is random and both indexed and non-indexed government bonds co-exist, or as one where there is a hierarchy of claims against the government, and bonds bearing high expected returns are subject to some risk of partial default. Thus the model can confront a number of ways in which governments borrow using bonds with random returns.

The following features of our model, and their relationship with recent research on randomization, are of some interest. First, in our model randomization is desirable even though agents have the same underlying utility functions and endowments. This contrasts with the results in Prescott-Townsend (1984) and Arnott-Stiglitz (1988), and is due to the fact that agents have access to (and make differential use of) different non-market activities. Thus agents' indirect utility functions differ in such a way that randomized allocations may be desirable. This insight is essentially the same as that in Benhabib, Rogerson, and Wright (1990). Second, a type of randomization consistent with the predictions of our model has been observed historically. This is of interest because Arnott and Stiglitz (1988) (among others) note that it is puzzling that randomization of contracts does not occur as frequently as theory suggests. The analysis of government debt contracts in our model may provide some insight into this puzzle.

Arnott and Stiglitz discuss six reasons why randomization might not be observed: (1) agents do not understand that randomization is optimal (i.e., they are only boundedly rational); (2) contracts involving randomization are costly to enforce; (3) secondary markets or randomization insurance neutralize the effects of randomization; (4) lotteries are viewed as unfair; (5) von Neumann-Morgenstern expected utility theory is deficient; and (6) individuals do not trust randomization mechanisms. Our theory and historical observations suggest that (1), (2), (5), and (6) are not persuasive. We also regard (4) as unpersuasive because randomization of the type that we describe (i.e., weakening the voluntary participation constraint by treating type 1's preferentially rather than imposing randomization on the relatively disadvantaged type 2's) need not be perceived as unfair by either group.[16] In contrast, (3) may be an important reason that randomization (even of the type we describe) has been only periodic throughout history.[17]

Because the bond policy that we describe is a form of price discrimination, the existence of secondary markets would render the government unable to implement its program. That is, implementation of the constrained Pareto efficient bond policy that we study requires the government to impose legal restrictions that prohibit the intermediation of bonds with randomized returns. It is interesting to note that poorly developed financial markets are common in many high inflation countries that choose to monetize their deficits. For simplicity, we assume an ex cathedra prohibition against the intermediation of assets. However, Bencivenga and Smith (1991) study the optimal degree of financial repression in a developing economy faced with a sustained deficit that must be monetized. They find that a government with a deficit (that is either unwilling or unable to decrease spending or increase

[16]For example, if bonds with randomized returns were sold to foreign investors, it is unlikely that domestic investors would view the implicit taxation of foreign investors as unfair.

[17]Of course the size of g and x (see the restrictions associated with case 2 in Section 4 that make this policy optimal) also vary over the course of an economy's business cycle.

explicit taxes) may be required by simple feasibility to engage in financial repression to support its monetization program. Such repression is much more difficult in more developed countries and may be one reason why "lottery bonds" have not been observed in well developed financial markets.

Finally, another reason that the type of randomization policy we study has been observed only periodically throughout history may be related to the question of whether the policy supports a unique stationary equilibrium. An open question is whether the method of decentralization that we describe, based on Bryant and Wallace (1984), also supports other equilibria. While this must remain a topic for future research, Cooley and Smith (1990) have shown that the decentralization scheme in the Bryant and Wallace model can result in indeterminacies. In addition, the kinds of government borrowing schemes we describe may easily permit stationary sunspot equilibria, such as those described by Shell (1977), Azariadis (1981), and Cass and Shell (1983) [see Smith (1989) for a suggestive example along these lines]. Thus the market might add extraneous uncertainty to that already created by the government. These possibilities raise an interesting question for future research—how well can the government do, in a welfare sense, if it is constrained to schemes that have a unique (or a unique stationary) equilibrium?

7 Appendix A: Proof of Proposition 1

We first restate Problem 3.3 with only the binding constraints displayed.

Problem 3.3.$'$ *For $i = 1, 2$, choose c_1^i, c_2^i, and k to maximize*

$$\sum_{i=1}^{2} \theta_i [u(c_1^i) + v(c_2^i)]$$

subject to

$$\sum_{i=1}^{2} \theta_i (c_1^i + c_2^i) + \theta_1 k \leq w_1 + w_2 - g + \theta_1 x k \tag{A.1}$$

$$u(c_1^1) + v(c_2^1) \geq V(w_1, w_2, x) \tag{A.2}$$

$$u(c_1^2) + v(c_2^2) \geq u(c_1^1 + k) + v(c_2^1 - xk). \tag{A.3}$$

Proof of Proposition 1. Let $\lambda_n \geq 0$ be the Lagrange multiplier associated with constraint $(A.n)$, and observe the following.

At interior solutions for c_1^1 and c_2^1, the relevant first order conditions are

$$u'(c_1^1)(\theta_1 + \lambda_2) - \lambda_3 u'(c_1^1 + k) = \theta_1 \lambda_1. \tag{A.4}$$

$$v'(c_2^1)(\theta_1 + \lambda_2) - \lambda_3 v'(c_2^1 - xk) = \theta_1 \lambda_1. \tag{A.5}$$

The first order condition for k is

$$\lambda_3[v'(c_2^1 - xk)x - u'(c_1^1 + k)] - \lambda_1\theta_1(1 - x) \leq 0, \tag{A.6}$$

with equality if $k > 0$.

Finally, the first order conditions for c_1^2 and c_2^2 at an interior optimum are

$$u'(c_1^2)(\theta_2 + \lambda_3) = \theta_2 \lambda_1. \tag{A.7}$$

$$v'(c_2^2)(\theta_2 + \lambda_3) = \theta_2 \lambda_1. \tag{A.8}$$

Of course $(A.7)$ and $(A.8)$ imply that $u'(c_1^2) = v'(c_2^2)$.

Now multiply both sides of $(A.5)$ by x, subtract the result from $(A.4)$, and use $(A.6)$ to eliminate $\theta_1\lambda_1(1 - x)$ to obtain

$$(\theta_1 + \lambda_2)[xv'(c_2^1) - u'(c_1^1)] \leq 0, \tag{A.9}$$

with equality if $k > 0$.

Clearly, if we establish that $k > 0$ the proof is complete. Thus, suppose by way of contradiction that $k = 0$. Then $(A.4)$ and $(A.5)$ imply that $u'(c_1^1) = v'(c_2^1)$. Since $(A.2)$ is binding, it follows that $c_j^1 > c_j^2$, for $j = 1, 2$. But then $(A.3)$ is violated, giving the desired result. This proves Proposition 1.

8 Appendix B: Proof of Proposition 2

We begin by considering the following augmented version of Problem 4.1.

Problem 4.1′: *For $i = 1, 2$ and $s = 1, 2$, choose c_1^i, $c_2^i(s)$, and k to maximize*

$$\sum_{i=1}^{2} \theta_i \{u(c_1^i) + pv(c_2^i(1)) + (1-p)v(c_2^i(2))\}$$

subject to: (9) *through* (11) *and*

$$u'(c_1^1) = xpv'(c_2^1(1)) + x(1-p)v'(c_2^1(2)). \tag{B.1}$$

Since the solution to Problem 4.1 satisfies $(B.1)$, imposition of this constraint does not alter the optimal choices for the social planner.

Equations (9) through (11), which hold as equalities, and $(B.1)$ constitute four equations involving c_1^1, $c_2^1(1)$, $c_2^1(2)$, k, c_1^2, and c_2^2 [since $c_2^2(1) = c_2^2(2) = c_2^2$]. We now use (9), (11), and $(B.1)$ to eliminate c_1^1, $c_2^1(1)$, and k from Problem 4.1′.

First, let $(B.1)$ define c_1^1 as a function of $c_2^1(1)$ and $c_2^1(2)$. In particular, define $c_1^1 \equiv \alpha(c_2^1(1), c_2^1(2))$. Clearly, $\tilde{c}_1^1 = \alpha(\tilde{c}_2^1, \tilde{c}_2^1)$ holds. In addition, differentiation of $(B.1)$ yields

$$\alpha_1(c_2^1(1), c_2^1(2)) = px\frac{v''(c_2^1(1))}{u''(c_1^1)} > 0. \tag{B.2}$$

$$\alpha_2(c_2^1(1), c_2^1(2)) = (1-p)x\frac{v''(c_2^1(2))}{u''(c_1^1)} > 0. \tag{B.3}$$

Second, substitute $c_1^1 = \alpha(c_2^1(1), c_2^1(2))$ into (9) at equality. This gives k as a function of $c_2^1(1)$, $c_2^1(2)$, c_1^2, and c_2^2. Thus define $k \equiv \beta(c_2^1(1), c_2^1(2); c_1^2, c_2^2)$. Observe that $\tilde{k} = \beta(\tilde{c}_2^1, \tilde{c}_2^1; \tilde{c}_1^2, \tilde{c}_2^2)$ holds, and that differentiation of $\beta(\cdot)$ yields:

$$\beta_1 = -(\frac{\alpha_1 + p}{1-x}) < 0. \tag{B.4}$$

$$\beta_2 = -(\frac{\alpha_2 + 1 - p}{1 - x}) < 0. \quad (B.5)$$

Third, substitute $c_1^1 = \alpha(c_2^1(1), c_2^1(2))$ and $k = \beta(\cdot)$ into (11) at equality. This defines $c_2^1(2)$ as a function of $c_2^1(1)$, c_1^2, and c_2^2; say $c_2^1(2) \equiv \gamma(c_2^1(1); c_1^2, c_2^2)$. As before $\tilde{c}_2^1 = \gamma(\tilde{c}_2^1; \tilde{c}_1^2, \tilde{c}_2^2)$. Moreover, differentiation of $\gamma(\cdot)$ yields

$$\gamma_1(\tilde{c}_2^1; \tilde{c}_1^2, \tilde{c}_2^2) = -\frac{p}{1-p}. \quad (B.6)$$

Finally, define the function $\delta(\cdot)$ as follows

$$\delta(c_2^1(1); c_1^2, c_2^2) \equiv u(\alpha(c_2^1(1), \gamma(\cdot))) + pv(c_2^1(1)) + (1-p)v(\gamma(\cdot)). \quad (B.7)$$

Observe that $\delta(\cdot)$ is the left-hand-side of constraint (10), the (binding) voluntary participation constraint for type 1 agents. The function $\delta(\cdot)$ expresses the left-hand-side of this constraint solely as a function of $c_2^1(1)$ and c_j^2, for $j = 1, 2$. The strategy of the remainder of the proof is to show that $\delta(\cdot)$ is locally convex in $c_2^1(1)$, so local randomization relaxes the voluntary participation constraint on type 1 agents and consequently is welfare improving.

Now observe that Problem 4.1′ reduces to the following:[18]

Problem 4.1″. *Choose c_1^2, c_2^2, and $c_2^1(1)$ to maximize*

$$u(c_1^2) + v(c_2^2)$$

subject to:

$$\delta(c_2^1(1); c_1^2, c_2^2) \geq V(w_1, w_2, x). \quad (B.8)$$

If $c_2^1(1) = \tilde{c}_2^1$ at an optimum, then the solution to Problem 4.1″ coincides with the (non-stochastic) solution to Problem 3.3.

We now establish the following properties of $\delta(\cdot)$:

$$\delta_1(\tilde{c}_2^1; \tilde{c}_1^2, \tilde{c}_2^2) = 0, \quad (B.9)$$

[18]This follows since $u(c_1^1) + pv(c_2^1(1)) + (1-p)v(c_2^1(2)) = V(w_1, w_2, x)$ holds.

and if the inequality in Proposition 2 holds,

$$\delta_{11}(\tilde{c}_2^1; \tilde{c}_1^2, \tilde{c}_2^2) > 0. \qquad (B.10)$$

Then setting $c_2^1(1) \neq \tilde{c}_2^1$ (in some neighborhood of $\tilde{c}_2^1$) relaxes constraint (B.7) in Problem 4.1″. It follows that at an optimum, $c_2^1(1) \neq \tilde{c}_2^1$, and consequently $c_2^1(1) \neq c_2^1(2)$. Thus there will be extraneous randomization of the allocation received by type 1 agents.

It remains, then, to establish that (B.9) and (B.10) hold. For (B.9), straightforward differentiation of (B.7) gives

$$\delta_1(\tilde{c}_2^1; \tilde{c}_1^2, \tilde{c}_2^2) = u'(\alpha(\cdot))[\alpha_1 + \alpha_2\gamma_1] + pv'(\tilde{c}_2^1) + (1-p)v'(\tilde{c}_2^1)\gamma_1. \qquad (B.11)$$

Substitution of (B.2), (B.3), and (B.6) into (B.11) gives (B.9)

For (B.10), further differentiation yields

$$\delta_{11}(\tilde{c}_2^1; \tilde{c}_1^2, \tilde{c}_2^2) = u''(\tilde{c}_1^1)[\alpha_1+\alpha_2\gamma_1]^2+u'(\tilde{c}_1^1)[\alpha_{11}+\alpha_{12}\gamma_1+\alpha_{21}\gamma_1+\alpha_{22}(\gamma_1)^2+\alpha_2\gamma_{11}]$$

$$+pv''(\tilde{c}_2^1) + (1-p)v''(\tilde{c}_2^1)(\gamma_1)^2 + (1-p)v'(\tilde{c}_2^1)\gamma_{11}. \qquad (B.12)$$

It is straightforward but tedious to establish that, when evaluated at ($\tilde{c}_2^1$, $\tilde{c}_1^2$, $\tilde{c}_2^2$),

$$\alpha_{11} + \alpha_{12}\gamma_1 + \alpha_{21}\gamma_1 + \alpha_{22}(\gamma_1)^2 + \alpha_2\gamma_{11} =$$

$$\frac{1-p}{p}\alpha_1\{\gamma_{11} + [\frac{p}{(1-p)^2}]\frac{v'''(\tilde{c}_2^1)}{v''(\tilde{c}_2^1)}\}; \qquad (B.13)$$

$$\alpha_1 + \alpha_2\gamma_1 = 0; \qquad (B.14)$$

and

$$\gamma_{11}(\tilde{c}_2^1; \tilde{c}_1^2, \tilde{c}_2^2)(p + x\alpha_1) = -\frac{px\alpha_1 v'''(\tilde{c}_2^1)}{(1-p)^2 v''(\tilde{c}_2^1)} -$$

$$\frac{p^2 v''(\tilde{c}_2^1 - x\tilde{k})(1-x)}{(1-p)^2[v'(\tilde{c}_2^1 - x\tilde{k}) - u'(\tilde{c}_1^1 + \tilde{k})]}. \qquad (B.15)$$

Substituting (B.13) through (B.15) into (B.12) gives

$$\delta_{11}(\tilde{c}_2^1;\tilde{c}_1^2,\tilde{c}_2^2) = \frac{pv''(\tilde{c}_2^1)}{1-p} + \frac{x\alpha_1 v'(\tilde{c}_2^1)v'''(\tilde{c}_2^1)}{(1-p)v''(\tilde{c}_2^1)} + (1-p)v'(\tilde{c}_2^1)\gamma_{11}[1+\frac{x\alpha_1}{p}]$$

$$= \frac{pv''(\tilde{c}_2^1)}{1-p} - \frac{v'(\tilde{c}_2^1)p(1-x)v''(\tilde{c}_2^1 - x\tilde{k})}{(1-p)[v'(\tilde{c}_2^1 - x\tilde{k}) - u'(\tilde{c}_1^1 + \tilde{k})]}. \qquad (B.16)$$

Clearly, $\delta_{11}(\tilde{c}_2^1;\tilde{c}_1^2,\tilde{c}_2^2) > 0$ if the inequality in Proposition 2 holds. This completes the proof.

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